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PERIODIC SOLUTIONS OF SETS OF ORDINARY DIFFERENTIAL EQUATIONS WITH A LARGE PARAMETER*

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The differential equation of the forced oscillations of a mechanical system with one degree of freedom is examined in the case when the system's natural frequency is much greater than the external one. It is shown that periodic solutions of such an equation exist, close to the periodic solutions of the corresponding degenerate equation. The result obtained is generalized to the case of systems with several degrees of freedom. A system with cyclic coordinates acted on by external periodic forces whose frequency is much less than the natural frequencies of the system mentioned is examined. The existence of periodic solutions of the equations of motion of such a system, close to the periodic solutions of the corresponding degenerate equations, is proved.

1. Consider the scalar differential equation

$$x^{\cdot\cdot} + \mu^2 F(t, x) = f(t, x, x^{\cdot}) \tag{1.1}$$

where μ is a positive parameter, and $F(t,x)$ and $f(t,x,x^{\cdot})$ are periodic functions of t with period $T>0$. Let the equation $F(t,x)=0$ have the T-periodic solution $x=\varphi(t)$. We will seek T-periodic solutions of Eq.(1.1), defined for fairly large μ and close to the solution $x=\varphi(t)$. We will assume that the functions $F(t,x), f(t,x,x^{\cdot}), \varphi(t)$ are thrice continuously differentiable for $0\leqslant t\leqslant T$ and fairly small $|x-\varphi(t)|, |x^{\cdot}-\varphi^{\cdot}(t)|$ and

$$p(t) = \partial F(t, \varphi(t))/\partial x > 0 \quad (0\leqslant t\leqslant T)$$

Let

$$a = -\frac{1}{4b}\int_0^T \frac{\partial f(t, \varphi(t), \varphi^{\cdot}(t))}{\partial x^{\cdot}}\,dt, \qquad b = \frac{1}{2}\int_0^T p^{1/2}(t)\,dt$$

$$\varepsilon_0 = \sqrt{1+\operatorname{sh}^2 ab}$$

For an arbitrary $\varepsilon\in(0,\varepsilon_0)$ we consider the set

$$I(\varepsilon) = \{\mu : \mu>0,\ \operatorname{sh}^2 ab + \sin^2\mu b > \varepsilon^2\}$$

This set is not empty. For $a\neq 0$ and $0<\varepsilon<|\operatorname{sh} ab|$ it is identical with the interval $(0,+\infty)$, and when $a=0$

$$I(\varepsilon) = \bigcup_{k=1}^{\infty}\left[\frac{\pi(k-1)+\arcsin\varepsilon}{b}, \frac{\pi k-\arcsin\varepsilon}{b}\right]$$

Theorem 1. For any $\varepsilon\in(0,\varepsilon_0)$ positive numbers M, C_1 and C_2 exist such that for $\mu\geqslant M$; $\mu\in I(\varepsilon)$, Eq.(1.1) has a unique T-periodic solution $x_*(t,\mu)$ satisfying the inequalities

$$|x_*(t,\mu)-\varphi(t)|\leqslant\frac{C_1}{\mu^2}, \quad |x_*^{\cdot}(t,\mu)-\varphi^{\cdot}(t)|\leqslant\frac{C_2}{\mu}\ (0\leqslant t\leqslant T)$$

Note. If $a\neq 0$, the quantities M, C_1 and C_2 can be chosen independently of ε (but then $M, C_1, C_2\to+\infty$ as $a\to 0$). Having taken $\varepsilon<|\operatorname{sh} ab|$, we find that in this case the solution $x_*(t,\mu)$ is defined for any $\mu\geqslant M$. If $p(t)<0$ $(0\leqslant t\leqslant T)$, then the existence of a T-periodic solution of Eq.(1.1), changing to $\varphi(t)$ as $\mu\to+\infty$, follows from the results in /1/. The case when $p(t)$ vanishes at some points of the segment $[0,T]$ requires a special investigation.

Equation (1.1) can be interpreted as the equation of forced oscillations of a mechanical system with one degree of freedom. The quantity of $2a$ can be taken as a generalized coefficient of friction of this system for the motion $x\approx\varphi(t)$. Resonance is possible in the system for certain values of μ. Such values are excluded from the analysis by the condition $\mu\in I(\varepsilon)$. If $a\neq 0$, this condition can be dropped by examining sufficiently large values of μ. However, if $a=0$, resonance can occur in the system on any segment of the μ-axis, of length greater than π/b, sufficiently far from the origin.

For example, the oscillations around the centre of mass of a solid with a strong permanent magnet in the external periodic magnetic field /2/ can be described by an equation of

*Prikl.Matem.Mekhan., 47, 5, 707-719, 1983 form (1.1). For the equation studied in /2/, $a=0$. The results of the numerical calculations in this paper graphically demonstrate the occurrence of resonances for certain values of the large parameter.

2. To prove Theorem 1 we make the change of variable $(t, x) \to (\tau, y)$ in Eq. (1.1):

$$\tau = \int_0^t p^{1/2}(s)\,ds, \qquad y = [x - \varphi(\psi(\tau))]\exp\left(\frac{1}{2}\int_0^\tau c(s)\,ds - a\tau\right)$$

$$c(\tau) = \left[\frac{p^{\cdot}(t)}{2p^{3/2}(t)} - \frac{1}{p^{1/2}(t)}\frac{\partial f(t, \varphi(t), \varphi^{\cdot}(t))}{\partial x^{\cdot}}\right]_{t=\psi(\tau)}$$

where $t = \psi(\tau)$ is the inverse of the first integral. Such a change is a modified Liouville substitution. In the new variables (1.1) can be written as

$$y'' + 2ay' + (\mu^2 + a^2)\, y = f_1(\tau, y, y') + \mu^2 F_1(\tau, y) \tag{2.1}$$

Here the prime denotes differentiation with respect to τ; the functions f_1 and F_1 depend periodically on τ with period $2b$

$$\frac{\partial f_1(\tau, 0, 0)}{\partial y'} = F_1(\tau, 0) = \frac{\partial F_1(\tau, 0)}{\partial y} = 0 \tag{2.2}$$

The above change of variable reduces the search for a T-periodic solution of Eq. (1.1), close to $\varphi(t)$, to a search for a $2b$-periodic solution of Eq.(1.1), close to the origin.

By virtue of the smoothness conditions imposed on the functions F, f and φ, the functions f_1 and F_1 are continuously differentiable in τ and thrice continuously differentiable in y and y'. Hence it follows from (2.2) that positive numbers h_1, h_2, M_1, M_2 and M_3 exist such that the bounds

$$\begin{aligned} |f_1(\tau, y, y') - f_1(\tau, u, u')| &\leqslant M_1 |y - u| + \\ &M_2 |y' - u'| (|y| + |u| + |y'| + |u'|) \\ |F_1(\tau, y) - F_1(\tau, u)| &\leqslant M_3 |y - u| (|y| + |u|) \end{aligned} \tag{2.3}$$

are valid for all τ, y, y', u, u' satisfying the inequalities $0 \leqslant \tau \leqslant 2b$, $|y| \leqslant h_1$, $|u| \leqslant h_1$, $|y'| \leqslant h_2$, $|u'| \leqslant h_2$. In particular, when $u = u' = 0$ and $M_4 = M_1 + M_2 h_2$ we have

$$\begin{aligned} |f_1(\tau, y, y') - f_1(\tau, 0, 0)| &\leqslant M_4 |y| + M_2 y'^2 \\ |F_1(\tau, y)| &\leqslant M_3 y^2 \end{aligned} \tag{2.4}$$

3. Consider the differential equation

$$y'' + 2ay' + (\mu^2 + a^2)\, y = h(\tau) \tag{3.1}$$

where $h(\tau)$ is a continuously differentiable $2b$-periodic function. When $\operatorname{sh}^2 ab + \sin^2 \mu b > 0$ Eq. (3.1) has a unique $2b$-periodic solution representable as

$$y(\tau) = \int_0^{2b} G(\tau, s)\, h(s)\, ds = \frac{h(\tau)}{\lambda_1\lambda_2} + \int_0^{2b} G_1(\tau, s)\, h'(s)\, ds \tag{3.2}$$

$$G(\tau, s) = -\frac{\exp[\lambda_1(\tau - s \pm b)]}{2(\lambda_1 - \lambda_2)\operatorname{sh}\lambda_1 b} - \frac{\exp[\lambda_2(\tau - s \pm b)]}{2(\lambda_2 - \lambda_1)\operatorname{sh}\lambda_2 b}$$

$$G_1(\tau, s) = -\frac{\exp[\lambda_1(\tau - s + b)]}{2\lambda_1(\lambda_1 - \lambda_2)\operatorname{sh}\lambda_1 b} - \frac{\exp[\lambda_2(\tau - s \pm b)]}{2\lambda_2(\lambda_2 - \lambda_1)\operatorname{sh}\lambda_2 b}$$

$$\lambda_{1,2} = -a \pm i\mu, \quad i^2 = -1$$

Here $G(\tau, s)$ is Green's function of the periodic boundary-value problem $y(0) = y(2b)$, $y'(0) = y'(2b)$ for (3.1); in the expressions for $G(\tau, s)$ and $G_1(\tau, s)$ the upper sign is taken for $\tau \leqslant s$ and the lower one for $\tau > s$. The derivative of solution (3.2) can be found by the formula

$$y'(\tau) = \int_0^{2b} \frac{\partial G(\tau, s)}{\partial \tau}\, h(s)\, ds = \int_0^{2b} G(\tau, s)\, h'(s)\, ds$$

The number $\nu(f) = \max |f(\tau)|$ with $0 \leqslant \tau \leqslant 2b$ is called the norm of the function $f(\tau)$, continuous on the segment $[0, 2b]$. Since

$$\max \int_0^{2b} |G(\tau, s)|\, ds \leqslant K, \qquad \max \int_0^{2b} \left|\frac{\partial G(\tau, s)}{\partial \tau}\right| ds \leqslant K\sqrt{a^2 + \mu^2}$$

$$\max \int_0^{2b} |G_1(\tau, s)|\, ds \leqslant \frac{K}{\sqrt{a^2 + \mu^2}}; \qquad 0 \leqslant \tau \leqslant 2b$$

$$K = \frac{\operatorname{sh} ab}{a\mu\sqrt{\operatorname{sh}^2 ab + \sin^2 \mu b}}$$

the bounds

$$\begin{gathered}\nu(y) \leqslant K\nu(h), \quad \nu(y') \leqslant K\sqrt{a^2+\mu^2}\,\nu(h) \\ \nu(y) \leqslant \frac{\nu(h)}{a^2+\mu^2} + \frac{K\nu(h')}{\sqrt{a^2+\mu^2}}, \quad \nu(y') \leqslant K\nu(h')\end{gathered} \tag{3.3}$$

hold for the norms of solution (3.2) and of its derivative, valid both when $a \neq 0$ as well as when $a = 0$. In the latter case the values of the coefficients containing a are found by passing to the limit as $a \to 0$. In particular, when $a = 0$ we have $K = b/(\mu\,|\sin \mu b|)$.

We fix an arbitrary $\varepsilon \in (0, \varepsilon_0)$ and we set $N = \operatorname{sh} ab/a\varepsilon$. Then, by sharpening inequalities (3.3), for $\mu \in I(\varepsilon)$, $\mu \geqslant |a|$, we can write

$$\nu(y) \leqslant N\mu^{-1}\nu(h), \quad \nu(y') \leqslant 2N\nu(h) \tag{3.4}$$

$$\nu(y) \leqslant \mu^{-2}[\nu(h) + N\nu(h')], \quad \nu(y') \leqslant N\mu^{-1}\nu(h') \tag{3.5}$$

The resultant inequalities are meaningful for any a: if $a = 0$, then $N = b/\varepsilon$. However, if $a \neq 0$, then we can take $N = |a|^{-1}$. In this case inequalities (3.4) and (3.5) hold for any $\mu \geqslant |a|$. In Section 4 the bounds (3.4), (3.5) are used without additional stipulations on the method of defining N and choosing μ.

4. The search for $2b$-periodic solutions of Eq. (2.1) reduces to solving the periodic boundary-value problem for this equation on the segment $[0, 2b]$, which in its turn is equivalent to the system of integral equations

$$y_j(\tau) = \int_0^{2b} g_j(\tau, s)\,[f_1(s, y_1(s), y_2(s)) + \mu^2 F_1(s, y_1(s))]\,ds \equiv L_j(y_1, y_2) \tag{4.1}$$

$$j = 1,2; \quad g_1(\tau, s) = G(\tau, s), \quad g_2(\tau, s) = \partial G(\tau, s)/\partial\tau$$

Here $y_1 = y$, $y_2 = y'$. We solve system (4.1) by the method of successive approximations. On the segment $0 \leqslant \tau \leqslant 2b$ we construct a sequence of functions $\{y_j^{(k)}(\tau)\}_{k=0}^{\infty}$ $(j = 1, 2)$, by setting

$$y_j^{(0)}(\tau) \equiv 0, \quad y_j^{(k+1)} = L_j(y_1^{(k)}, y_2^{(k)}) \quad (j = 1, 2; \ k = 0, 1, \ldots) \tag{4.2}$$

Let us prove that when μ is sufficiently large this sequence converges (in the sense of the norm $\nu(\cdot)$) to the solution of system (4.1). First we will prove that for sufficiently large μ

$$\nu(y_1^{(k)}) \leqslant B_1\mu^{-2} \leqslant h_1, \quad \nu(y_2^{(k)}) \leqslant B_2\mu^{-1} \leqslant h_2 \quad (k = 0, 1, \ldots) \tag{4.3}$$

where B_1 and B_2 are certain positive numbers. Since

$$y_j^{(1)}(\tau) = \int_0^{2b} g_j(\tau, s) f_1(s, 0, 0)\,ds \quad (j = 1, 2) \tag{4.4}$$

relations (4.2) for $k \geqslant 1$ can be represented as

$$y_j^{(k+1)}(\tau) = y_j^{(1)}(\tau) + \int_0^{2b} g_j(\tau, s)\,[f_1(s, y_1^{(k)}(s), y_2^{(k)}(s)) - f_1(s, 0, 0) + \mu^2 F_1(s, y_1^{(k)}(s))]\,ds \quad (j = 1, 2)$$

We assume that $\nu(y_j^{(k)}) \leqslant h_j$ $(j = 1, 2;\ k = 0, 1, \ldots)$. Then on the strength of inequalities (2.4) and (3.4) we have

$$\begin{gathered}\nu(y_1^{(1)}) \leqslant D_1\mu^{-2}, \quad \nu(y_2^{(1)}) \leqslant D_2\mu^{-1} \\ D_2 = N\nu(\partial f_1(\tau, 0, 0)/\partial\tau), \quad D_1 = D_2 + \nu(f_1(\tau, 0, 0))\end{gathered} \tag{4.5}$$

With the aid of relations (4.4) and (4.5) we obtain the bounds

$$\nu(y_j^{(k+1)}) \leqslant \nu(y_j^{(1)}) + n_j[M_4\nu(y_1^{(k)}) + \mu^2 M_3\nu^2(y_1^{(k)}) + M_2\nu^2(y_2^{(k)})]$$

$$j = 1, 2; \ n_1 = N\mu^{-1}, \ n_2 = 2N$$

We choose the numbers B_1, B_2 from the conditions $B_1 > D_1$, $B_2 > D_2$ and we set

$$\mu \geqslant \mu_1 = \max\left(\sqrt{\frac{B_1}{h_1}}, \frac{B_2}{h_2}, \frac{\varkappa}{B_1 - D_1}, \frac{2\varkappa}{B_2 - D_2}\right)$$

$$\varkappa = N(M_4B_1 + M_3B_1^2 + M_2B_2^2)$$

Then if inequalities (4.3) are satisfied for some k, then by virtue of (4.5) we have

$$\nu(y_1^{(k+1)}) \leqslant \mu^{-2}(D_1 + \varkappa\mu^{-1}) \leqslant B_1\mu^{-2} \leqslant h_1$$

$$\nu(y_2^{(k+1)}) \leqslant \mu^{-1}(D_2 + 2\varkappa\mu^{-1}) \leqslant B_2\mu^{-1} \leqslant h_2$$

Since inequalities (4.3) are satisfied when $k=1$, their validity follows for all k.

Let us prove that the successive approximations (4.2) converge. On the strength of inequalities (2.3) and (3.4) we have

$$\nu(y_j^{(k+1)}-y_j^{(k)})\leqslant n_j[K_1\nu(y_1^{(k)}-y_1^{(k-1)})+K_2\nu(y_2^{(k)}-y_2^{(k-1)})]$$
$$j=1,2;\quad K_1=M_1+\mu^2M_3[\nu(y_1^{(k)})+\nu(y_1^{(k-1)})]$$
$$K_2=M_2[\nu(y_1^{(k)})+\nu(y_2^{(k)})+\nu(y_1^{(k-1)})+\nu(y_2^{(k-1)})]$$

Estimating K_1 and K_2 using inequalities (4.3), we obtain

$$\nu(y_j^{(k+1)}-y_j^{(k)})\leqslant n_j\rho_k\quad(j=1,2)$$
$$\rho_k=H_1\nu(y_1^{(k)}-y_1^{(k-1)})+H_2\mu^{-1}\nu(y_2^{(k)}-y_2^{(k-1)})$$
$$H_1=M_1+2M_3B_1,\quad H_2=2M_2(B_2+B_1\mu^{-1})\tag{4.6}$$

Consider the number sequence $\rho_k\ (k=1,2,\ldots)$. The relation

$$\rho_{k+1}\leqslant N\mu^{-1}(H_1+2H_2)\rho_k\quad(k=1,2,\ldots)$$

where $N\mu^{-1}(H_1+2H_2)\to 0$ as $\mu\to+\infty$, holds by virtue of (4.6). Therefore, a number $\mu_2>0$ exists such that the inequality $N\mu^{-1}(H_1+2H_2)\leqslant{}^1/_2$ holds when $\mu\geqslant\mu_2$. Let $\mu\geqslant M=\max(\mu_1,\mu_2)$. Then $\rho_{k+1}\leqslant\rho_k/2\ (k=1,2,\ldots)$. Using this bound it can be proved that the sequences $\{y_j^{(k)}(\tau)\}_{k=0}^{\infty}$ $(j=1,2)$ converge uniformly on the segment $0\leqslant\tau\leqslant 2b$ to some continuous functions $y_j^*(\tau)$. Since by the construction of these sequences $y_j^{(k)}(0)=y_j^{(k)}(2b)\ (j=1,2;\ k=0,1,\ldots)$, we have $y_j^*(0)=y_j^*(2b)$. Analogously, by virtue of (4.3).

$$\nu(y_1^*)\leqslant B_1\mu^{-2},\quad \nu(y_2^*)\leqslant B_2\mu^{-1}\tag{4.7}$$

Passing to the limit in relations (4.2) as $k\to\infty$, we find that $y_1^*(\tau)$ and $y_2^*(\tau)$ is the solution of system (4.1), where the function $y_1^*(\tau)$ is twice continuously differentiable and $dy_1^*(\tau)/d\tau=y_2^*(\tau)$.

Let us prove the uniqueness of the solution found. Assume that system (4.1) has one more solution $y_1^0(\tau)$, $y_2^0(\tau)$ satisfying bounds (4.7). By the constructions described above, for the quantity $\rho=H_1\nu(y_1^*-y_1^\circ)+H_2\mu^{-1}\nu(y_2^*-y_2^\circ)$ we can establish the inequality $\rho\leqslant\rho/2$. Hence $\rho=0$ and, consequently, $y_j^\circ=y_j^*\ (j=1,2)$.

Continuing $y_1^*(\tau)$ along the whole real axis, by using the relations $y_1^*(\tau\pm 2b)=y_1^*(\tau)$, we obtain a $2b$-periodic solution of Eq. (2.1). The desired solution $x_*(t,\mu)$ of Eq. (1.1) corresponds to this solution. The validity of Theorem 1 is established by recalling the method of choosing μ, B_1, B_2, M and transforming the bounds (4.7) into bounds for $\max|x_*(t,\mu)-\varphi(t)|$, $\max|x_*^{\cdot}(t,\mu)-\varphi^{\cdot}(t)|$ for $0\leqslant t\leqslant T$.

5. Equation (1.1) was interpreted as the equation of motion of a mechanical system with one degree of freedom. In the remaining part of this paper an analogous problem is solved for a system with several degrees of freedom. Theorem 2 proved below is, to a certain extent, a generalization of Theorem 1.

We consider a mechanical system with l degrees of freedom, whose equations of motion can be written as

$$\frac{d}{dt}\frac{\partial T}{\partial q_j^{\cdot}}-\frac{\partial T}{\partial q_j}=-\mu^2\frac{\partial\Pi}{\partial q_j}+Q_j\quad(j=1,2,\ldots,l)\tag{5.1}$$

Here $q_1,\ldots,q_l$ are the system's generalized coordinates, μ is a positive parameter,

$$Q_j=Q_j(t,q_1,\ldots,q_l,q_1^{\cdot},\ldots,q_l^{\cdot})\quad(j=1,2,\ldots,l)\tag{5.2}$$

are generalized forces acting in the system,

$$\mu^2\Pi=\mu^2\Pi(q_1,\ldots,q_n)\tag{5.3}$$

is the system's potential energy, and

$$T=\frac{1}{2}\sum_{j,k=1}^{l}a_{jk}(q_1,\ldots,q_n)q_j^{\cdot}q_k^{\cdot}+\sum_{j=1}^{l}a_j(t,q_1,\ldots,q_l)q_j^{\cdot}+a_0(t,q_1,\ldots,q_l)\tag{5.4}$$

is its kinetic energy. We assume that in (5.4) the matrix $(a_{jk})_{j,k=1}^{l}$ does not contain t and is positive definite, the functions (5.2) and (5.4) depend 2π-periodically on t, and $1\leqslant n\leqslant l$ in (5.3) and (5.4). An example of a mechanical system described by Eqs. (5.1) is a magnetized solid moving around the centre of mass in a strong constant magnetic field and subject to the additional action of periodic external moments.

We seek periodic solutions of Eqs. (5.1), defined for fairly large μ. In order to formulate the problem precisely, in (5.1) we will change to Routh variables $q_j, q_j^{\cdot}, q_\alpha, p_\alpha = \partial T/\partial q_\alpha^{\cdot} (j = 1, 2, \ldots, n; \alpha = n+1, \ldots, l)$. These equations then take the form

$$\frac{d}{dt}\frac{\partial R}{\partial q_j^{\cdot}} - \frac{\partial R}{\partial q_j} = \mu^2 \frac{\partial \Pi}{\partial q_j} - Q_j \quad (j = 1, \ldots, n) \tag{5.5}$$
$$q_\alpha^{\cdot} = \frac{\partial R}{\partial p_\alpha}, \quad p_\alpha^{\cdot} = Q_\alpha \quad (\alpha = n+1, \ldots, l)$$

Here

$$R = \sum_{\alpha=n+1}^{l} p_\alpha q_\alpha^{\cdot} - T = -\frac{1}{2}\sum_{j,k=1}^{n} a_{jk}^0(q) q_j^{\cdot} q_k^{\cdot} + \sum_{j=1}^{n} b_j(t, x, q) q_j^{\cdot} + b_0(t, x, q)$$

$q = (q_1, \ldots, q_n)^T$, $x = (q_{n+1}, \ldots, q_l, p_{n+1}, \ldots, p_l)^T$, and the matrix $A_0(q) = (a_{jk}^0)_{j,k=1}^n$ is positive definite. By appropriately introducing the functions $F(t, x, q, q^{\cdot}) \in R^{2(l-n)}$ and $f(t, x, q, q^{\cdot}) \in R^n$, Eqs. (5.5) become

$$x^{\cdot} = F(t, x, q, q^{\cdot}) \tag{5.6}$$
$$A_0(q) q^{\cdot\cdot} + \mu^2 \frac{\partial \Pi(q)}{\partial q} = f(t, x, q, q^{\cdot})$$

The resultant system is of independent interest since the equations of motion of certain mechanical systems reduce to the form (5.6) without the use of Routh variables. Below we examine Eqs. (5.6) without relating it to Eqs. (5.5). We assume that in (5.6) x and $F \in R^m$ $(m \geqslant 0)$, q and $f \in R^n$ $(n \geqslant 1)$, $\Pi \in R^1$, F and f are 2π-periodically dependent on t, $A_0(q)$ is a symmetric positive-definite matrix of order n. The functions $\Pi(q), A_0(q), f(t, x, q, q^{\cdot})$ and $F(t, x, q, q^{\cdot})$ are taken to be fairly smooth functions of their arguments, i.e., have all the derivatives needed for subsequent analysis. We assume as well that $\partial\Pi(0)/\partial q = 0$ and that the matrix $\partial^2\Pi(0)/\partial q^2$ is positive definite.

The system

$$x^{\cdot} = F(t, x, 0, 0)$$

is called degenerate. Suppose this system has a 2π-periodic solution $x = \varphi(t)$. We seek the 2π-periodic solutions $x(t, \mu), q(t, \mu)$ of system (5.6), defined for values of μ from some unbounded set $I_\mu \subset (0, +\infty)$ and satisfying as $\mu \to +\infty$, $\mu \in I_\mu$ the relations $x(t, \mu) - \varphi(t) = O(\mu^{-1})$, $q(t, \mu) = O(\mu^{-2})$, $q^{\cdot}(t, \mu) = O(\mu^{-1})$. In system (5.6) there may not be equations for x $(m = 0)$. In this case we examine the system

$$A_0(q) q^{\cdot\cdot} + \mu^2 \frac{\partial \Pi(q)}{\partial q} = f(t, q, q^{\cdot}) \tag{5.7}$$

and seek its 2π-periodic solutions $q(t, \mu)$, which, as $\mu \to +\infty, \mu \in I_\mu$, satisfy the conditions $q(t, \mu) = O(\mu^{-2})$, $q^{\cdot}(t, \mu) = O(\mu^{-1})$. When $n = 1$ the existence of such solutions follow from Theorem 1. System (5.7) can be analyzed in the same was as system (5.6) (in the latter we must omit all sections referring to the vector x) and, therefore, we will not do so here.

6. To construct the periodic solutions of system (5.6) we transform it as follows. We make the change of variables $x = \varphi(t) + \xi$ and we multiply the second equation on the left by $A_0^{-1}(q)$. In the resulting equations we pick out in explicit form certain terms linear in ξ, q and $q^{\cdot}$. As a result we have

$$\xi^{\cdot} = A(t)\xi + F_1(t, \xi, q, q^{\cdot}) \tag{6.1}$$
$$q^{\cdot\cdot} + \mu^2\Lambda q = B(t)\xi + C(t) q^{\cdot} + f_1(t, \xi, q, q^{\cdot}) + \mu^2 h_1(q)$$

Here

$$A(t) = \frac{\partial F(t, \varphi(t), 0, 0)}{\partial x}, \quad B(t) = A_0^{-1}(0)\frac{\partial f(t, \varphi(t), 0, 0)}{\partial x}$$
$$C(t) = A_0^{-1}(0)\frac{\partial f(t, \varphi(t), 0, 0)}{\partial q^{\cdot}}, \quad \Lambda = A_0^{-1}(0)\frac{\partial^2\Pi(0)}{\partial q^2}$$

and the relations

$$\|F_1(t, \xi, q, q^{\cdot})\| = O(\|q\| + \|q^{\cdot}\| + \|\xi\|^2), \ \|h_1(q)\| = O(\|q\|^2)$$
$$\|f_1(t, \xi, q, q^{\cdot}) - f_1(t, 0, 0, 0)\| = O(\|q\| + \|q^{\cdot}\|^2 + \|\xi\|^2)$$

where $\|\cdot\|$ is the Euclidean norm, are valid as $\xi, q, q^{\cdot} \to 0$. Since the matrices $A_0(0)$ and $\partial^2\Pi(0)/\partial q^2$ are symmetric and positive definite, the quadratic forms corresponding to them can be simultaneously reduced to canonical form. More precisely, a non-singular matrix S exists such that

$$S^T A_0(0) S = E_n, \quad S^T \frac{\partial^2\Pi(0)}{\partial q^2} S = \operatorname{diag}(\omega_1^2 E_{n_1}, \ldots, \omega_r^2 E_{n_r}) \tag{6.2}$$

Here E_k is the unit matrix of order k, $n_j > 0$ $(j = 1, 2, \ldots, r)$, $n_1 + n_2 + \ldots + n_r = n$, $0 < \omega_1 <$

$\omega_2 < \ldots < \omega_r$. Having made in (6.1) the change of variable $q = Sq'$ and returning to the former notation, we will assume that the matrix Λ in this system is identical with the right-hand side of the second formula in (6.2).

The following transformations are usual when investigating differential equations with a large parameter /1,3/. The substitution

$$q = z + \mu^{-2}\Lambda^{-1}B(t)\xi$$

reduces system (6.1) to

$$\xi^{\cdot} = A(t)\xi + F_2(t, \xi, z, z^{\cdot}, \mu) \tag{6.3}$$

$$z^{\cdot\cdot} + \mu^2\Lambda z = C(t)z^{\cdot} + f_2(t, \xi, z, z^{\cdot}, \mu) + \mu^2 h_2(t, \xi, z, \mu)$$

where we have

$$\| F_2(t, \xi, z, z^{\cdot}, \mu)\| = O(\|z\| + \|z^{\cdot}\| + \|\xi\|^2 + \mu^{-2}\|\xi\|)$$

$$\| f_2(t, \xi, z, z^{\cdot}, \mu) - f_2^0(t, \mu)\| = O\left(\|z\| + \|z^{\cdot}\|^2 + \|\xi\|^2 + \frac{\|\xi\| + \|z^{\cdot}\|}{\mu^2}\right)$$

$$\| f_2^0(t, \mu)\| = O(1), \quad \| h_2(t, \xi, z, \mu)\| = O(\|z\|^2 + \mu^{-4}\|\xi\|^2)$$

as $\xi, z, z^{\cdot}, \mu^{-1} \to 0$. Here and below, for an arbitrary function $g(t, \xi, \ldots, \ldots, \mu)$ we use the notation $g^0(t, \mu) = g(t, 0, 0, 0, \mu)$. As a result of this substitution the term $B(t)\xi$ vanishes from the second equation of the system being investigated.

The next transformation serves to simplify the term $C(t)z^{\cdot}$. Instead of z we introduce the new variable

$$u = z + \mu^{-2}D(t)z^{\cdot} \tag{6.4}$$

where $D(t)$ is a 2π-periodic matrix. An explicit form for $D(t)$ is indicated below. Differentiating relation (6.4) twice with respect to t relative to system (6.3), we obtain

$$u^{\cdot} = -D\Lambda z + [E_n + \mu^{-2}(D^{\cdot} + DC)]z^{\cdot} + D(h_2 + \mu^{-2}f_2) \tag{6.5}$$

$$u^{\cdot\cdot} = -\mu^2\Lambda z + (C - D\Lambda)z^{\cdot} + f_2'(t, \xi, z, z^{\cdot}, \mu) + \mu^2 h_2 \tag{6.6}$$

The same bounds as for the function f_2 in (6.3) hold for the function f_2' in (6.6) as $\xi, z, z^{\cdot}, \mu^{-1} \to 0$. Having solved relations (6.4), (6.5) for z and $z^{\cdot}$ with due regard to the equality $h_2(t, \xi, z, \mu) = h_1(z) + O(\mu^{-2})$, we find

$$z = u - \mu^{-2}D\{u^{\cdot} + D[\Lambda u - h_1(u)]\} + O(\mu^{-4})$$

$$z^{\cdot} = u^{\cdot} + D[\Lambda u - h_1(u)] + O(\mu^{-2})$$

Substituting the resultant expressions into (6.6) and the first equation of (6.3), we arrive at the system

$$\xi^{\cdot} = A(t)\xi + F_3(t, \xi, u, u^{\cdot}, \mu) \tag{6.7}$$

$$u^{\cdot\cdot} + \mu^2\Lambda u = C'(t)u^{\cdot} + f_3(t, \xi, u, u^{\cdot}, \mu) + \mu^2 h_3(t, \xi, u, u^{\cdot}, \mu$$

Here

$$C'(t) = C(t) + \Lambda D(t) - D(t)\Lambda \tag{6.8}$$

The estimates

$$\| F_3(t, \xi, u, u^{\cdot}, \mu) - F_3^0(t, \mu)\| = O(\|u\| + \|u^{\cdot}\| + \|\xi\|^2 + \mu^{-2}\|\xi\|)$$

$$\| h_3(t, \xi, u, u^{\cdot}, \mu) - h_3^0(t, \mu)\| = O\left(\|u\|^2 + \frac{\|u\| + \|u^{\cdot}\|^2 + \|\xi\|^2}{\mu^4} + \frac{\|\xi\| + \|u^{\cdot}\|}{\mu^6}\right)$$

$$\| F_3^0(t, \mu)\| = O(\mu^{-2}), \quad \| h_3^0(t, \mu)\| = O(\mu^{-6})$$

hold for the functions F_3 and h_3 as $\xi, u, u^{\cdot}, \mu^{-1} \to 0$. The estimates for f_3 are obtained from those for f_2 by making the change $z \to u$, $z^{\cdot} \to u^{\cdot}$.

We will represent the matrices C, C' and D in block form, where the partitioning into blocks is the same as in the second formula of (6.2). Let $C = (C_{jk})_{j,k=1}^r$, $C' = (C_{jk}')_{j,k=1}^r$, $D = (D_{jk})_{j,k=1}^r$ where C_{jk}, C_{jk}' and D_{jk} are matrices of size $n_j \times n_k$. Then relation (6.8) can be written as

$$C_{jk}' = C_{jk} + (\omega_j^2 - \omega_k^2)D_{jk} \quad (j, k = 1, \ldots, r)$$

We define the matrix $D(t)$ as follows. We set $D_{jk} = (\omega_k^2 - \omega_j^2)^{-1}C_{jk}$ for $j \neq k$ and $D_{jj} = 0$. In this case $D(t + 2\pi) = D(t)$ and

$$C' = \mathrm{diag}(C_{11}, \ldots, C_{rr})$$

The last transformation that has to be made is to replace the 2π-periodic matrix $C'(t)$ in (6.7) by a constant matrix. Consider the matrix initial-value problems

$$X_j^{\cdot} = \frac{1}{2} C_{jj}(t) X_j, \quad X_j(0) = E_{n_j} \quad (j = 1, \ldots, r)$$

According to Floquet's theorem the solutions of these problems can be written in the form

$$X_j(t) = \Phi_j(t) \exp(H_j t), \quad H_j = \frac{1}{2\pi} \operatorname{Ln} X_j(2\pi)$$
$$\Phi_j(t + 2\pi) = \Phi_j(t) \quad (j = 1, \ldots, r)$$

If $X_j(2\pi)$ does not have negative eigenvalues, then the matrix H_j can be chosen to be real. Otherwise, this, in general, cannot be done. We can always choose a real matrix $H_j' = (4\pi)^{-1} \operatorname{Ln} X_j(4\pi)$, but then the matrix $\Phi_j'(t) = X_j(t) \exp(-H_j' t)$ satisfies the relation $\Phi_j'(t + 2\pi) = \Phi_j'(t) U_j$, where $U_j = X_j(2\pi) \exp(-2\pi H_j')$, $U_j^2 = E_{n_j}$. The use of such $\Phi_j'(t)$ in the tranformations that follow, somewhat complicates the investigation of the 2π-periodic solutions of system (5.6). As a result we have to solve either a periodic boundary-value problem in the interval $0 \leqslant t \leqslant 4\pi$ (because $\Phi_j'(t + 4\pi) = \Phi_j'(t)$ such an approach is simpler, but it can lead to a certain constriction on the set I_μ mentioned in Section 5) or a boundary-value problem in the interval $0 \leqslant t \leqslant 2\pi$, but not periodic, and with boundary conditions containing matrices U_j. Below, for brevity we examine the case when real $\operatorname{Ln} X_j(2\pi)$ $(j = 1, \ldots, r)$ exist. This occurs if, for example, system (5.6) is derived from Eqs. (5.1) in which the generalized forces (5.2) are potential forces.

As a matter of fact, without loss of generality we will assume that in (5.6) the matrices $A_0(0)$ and $\partial^2\Pi(0)/\partial q^2$ are identical with the right hand sides of formulas (6.2). Then in the case of potential forces (5.2) we must have $C(t) = -C^T(t)$ in (6.1). Hence $X_j^T(2\pi) = X_j^{-1}(2\pi)$ $(j = 1, \ldots, r)$, i.e., the matrices $X_j(2\pi)$ are orthogonal, and $\det X_j(2\pi) = 1$. The eigenvalues of matrices $X_j(2\pi)$ are located on the unit circle and have prime elementary divisors. The multiplicity of the eigenvalue -1 (if it exists) is even. In such a situation the matrices $\operatorname{Ln} X_j(2\pi)$ can always be chosen real. If the forces (5.2) are potential and $r = n$ in (6.2), then $C'(t) \equiv 0$. In this case $H_j = 0$, $\Phi_j(t) \equiv 1$ $(j = 1, \ldots, r)$.

We set

$$\Phi(t) = \operatorname{diag}(\Phi_1(t), \ldots, \Phi_r(t)), \quad H = \operatorname{diag}(H_1, \ldots, H_r)$$

The change of variable $u = \Phi(t) y$ converts (6.7) into the system

$$\xi^{\cdot} = A(t)\xi + F_4(t, \xi, y, y^{\cdot}, \mu) \tag{6.9}$$
$$y^{\cdot\cdot} - 2Hy^{\cdot} + (\mu^2\Lambda + H^2) y = f_4(t, \xi, y, y^{\cdot}, \mu) + \mu^2 h_4(t, \xi, y, y^{\cdot}, \mu)$$

As $\xi, y, y^{\cdot}, \mu^{-1} \to 0$ estimates hold for the functions F_4, f_4 and h_4, analogous to the estimates of the functions F_3, f_3 and h_3 as $\xi, u, u^{\cdot}, \mu^{-1} \to 0$. On the strength of these estimates positive numbers δ, K and μ_1 exist such that for all t, μ, ξ, η $(\eta \in R^m)$, y, $y^{\cdot}$, u, $u^{\cdot}$ satisfying the inequalities $0 \leqslant t \leqslant 2\pi$, $\mu \geqslant \mu_1$ and $\max(\|\xi\|, \|\eta\|, \|y\|, \|y^{\cdot}\|, \|u\|, \|u^{\cdot}\|) \leqslant \delta$ we have

$$\|F_4^0(t, \mu)\| \leqslant K\mu^{-2}, \quad \|f_4^0(t, \mu)\| \leqslant K, \|h_4^0(t, \mu)\| \leqslant K\mu^{-8} \tag{6.10}$$
$$\|F_4(t, \xi, y, y^{\cdot}, \mu) - F_4(t, \eta, u, u^{\cdot}, \mu)\| \leqslant K(\alpha_1 + \alpha_2 + \beta\alpha_0)$$
$$\|f_4(t, \xi, y, y^{\cdot}, \mu) - f_4(t, \eta, u, u^{\cdot}, \mu)\| \leqslant K[\alpha_1 + \beta(\alpha_0 + \alpha_2)]$$
$$\|h_4(t, \xi, y, y^{\cdot}, \mu) - h_4(t, \eta, u, u^{\cdot}, \mu)\| \leqslant$$
$$K\left(\alpha_1 + \frac{\alpha_0 + \alpha_2}{\mu^2}\right)\left(\|y\| + \|u\| + \frac{\|y^{\cdot}\| + \|u^{\cdot}\| + \|\xi\| + \|\eta\|}{\mu^2} + \frac{1}{\mu^4}\right)$$
$$\alpha_0 = \|\xi - \eta\|, \quad \alpha_1 = \|y - u\|, \quad \alpha_2 = \|y^{\cdot} - u^{\cdot}\|$$
$$\beta = \|\xi\| + \|\eta\| + \|y\| + \|u\| + \|y^{\cdot}\| + \|u^{\cdot}\| + \mu^{-2}$$

Having verified the transformations made above, we can convince ourselves that the independent variable t occurs 2π-periodically in system (6.9). The problem, posed in Section 5, of seeking 2π-periodic solutions of system (5.6) is equivalent to the problem of seeking 2π-periodic solutions $\xi(t, \mu)$, $y(t, \mu)$ of system (6.9), defined for values of μ from some unbounded set $I_\mu \subset (0, +\infty)$ and satisfying as $\mu \to +\infty$ $\mu \in I_\mu$, the relations $\xi(t, \mu) = O(\mu^{-1})$, $y(t, \mu) = O(\mu^{-2})$, $y^{\cdot}(t, \mu) = O(\mu^{-1})$.

Before we formulate the theorem for such solutions to exist we will introduce some definitions. Let P be a kth-order square matrix with eigenvalues $\lambda_1, \ldots, \lambda_k$. We introduce the function

$$\Delta(k, P, \mu) = |\det[\operatorname{sh}\pi(P + i\mu E_k)]| = \left\{\prod_{j=1}^{k} [\operatorname{sh}^2(\pi \operatorname{Re}\lambda_j) + \sin^2\pi(\mu + \operatorname{Im}\lambda_j)]\right\}^{1/2}$$

For an arbitrary $\varepsilon > 0$ we consider the set

$$I(\varepsilon) = \{\mu : \mu \geqslant 0, \ \Delta(n_j, H_j, \mu\omega_j) \geqslant \varepsilon \ (j = 1, \ldots, r)\}$$

Each function $\Delta(n_j, H_j, \mu\omega_j)$ is periodic in μ. Therefore, a number $\varepsilon_0 > 0$ exists such that for $0 < \varepsilon < \varepsilon_0$ either $I(\varepsilon) = [0, +\infty)$ or

$$I(\varepsilon) = \bigcup_{k=1}^{\infty} [\alpha_k(\varepsilon), \beta_k(\varepsilon)], \quad 0 \leqslant \alpha_1(\varepsilon) \leqslant \beta_1(\varepsilon) < \alpha_2(\varepsilon) \leqslant \beta_2(\varepsilon) < \ldots$$

$$0 < l_1 < \overline{\lim_{k\to\infty}} [\beta_k(\varepsilon) - \alpha_k(\varepsilon)] < l_2 < +\infty$$

and the numbers l_1 and l_2 can be chosen to be independent of ε. The first or the second of these possibilities is realized depending on whether all or not all eigenvalues of the matrices H_j $(j = 1, \ldots, r)$ have non-zero real parts.

Theorem 2. Assume that the system

$$\xi^{\cdot} = A(t)\,\xi \tag{6.11}$$

does not have non-zero 2π-periodic solutions. Then for any $\varepsilon \in (0, \varepsilon_0)$ positive numbers C_0, C_1, C_2 and M exist such that for $\mu \geqslant M$, $\mu \in I(\varepsilon)$, system (6.9) has a unique 2π-periodic solution $\xi_*(t, \mu)$, $y_*(t, \mu)$ satisfying the inequalities

$$\|\xi_*(t,\mu)\| \leqslant \frac{C_0}{\mu}, \quad \|y_*(t,\mu)\| \leqslant \frac{C_1}{\mu^2}$$

$$\|y_*^{\cdot}(t,\mu)\| \leqslant \frac{C_2}{\mu} \quad (0 \leqslant t \leqslant 2\pi)$$

7. In this section we derive certain relations which will be useful in proving Theorem 2. Let us consider the linear inhomogeneous system

$$\xi^{\cdot} = A(t)\,\xi + F(t) \tag{7.1}$$

where $F(t)$ is a 2π-periodic function, corresponding to the first equation in (6.9). By the hypothesis of Theorem 2 (the absence of non-trivial 2π-periodic solutions in (6.11) this system has the unique 2π-periodic solution

$$\xi(t) = \int_0^{2\pi} G_0(t,\tau) F(\tau)\, d\tau \tag{7.2}$$

Here $G_0(t,\tau)$ is Green's function for the periodic boundary-value problem $\xi(0) = \xi(2\pi)$ for (7.1).

The norm of the vector function $f(t)$ continuous in the interval $[0, 2\pi]$ is the number $\nu(f) = \max \|f(t)\|$ for $0 \leqslant t \leqslant 2\pi$. The estimate

$$\nu(\xi) \leqslant N_0 \nu(F) \tag{7.3}$$

where N_0 is some positive number, holds for the norm of solution (7.2).

Let us now consider the linear equation

$$y^{\cdot\cdot} - 2Hy^{\cdot} + (\mu^2\Lambda + H^2)\, y = f(t) \tag{7.4}$$

where $f(t)$ is a 2π-periodic function, corresponding to the second equation in system (6.9). If $\Delta(n_j, H_j, \mu\omega_j) > 0$ $(j = 1, \ldots, r)$, then this equation has the unique 2π-periodic solution

$$y(t) = \int_0^{2\pi} G(t,\tau) f(\tau)\, d\tau = (\Lambda_1\Lambda_2)^{-1} f(t) + \int_0^{2\pi} G'(t,\tau) f^{\cdot}(\tau)\, d\tau \tag{7.5}$$

$$G(t,\tau) = -\frac{1}{2}[(\Lambda_1 - \Lambda_2)\,\mathrm{sh}\,\pi\Lambda_1]^{-1} \exp[\Lambda_1(t - \tau \pm \pi)] + \mathrm{idem}\,(1 \leftrightarrow 2)$$

$$G'(t,\tau) = -\frac{1}{2}[(\Lambda_1 - \Lambda_2)\,\Lambda_1\,\mathrm{sh}\,\pi\Lambda_1]^{-1} \exp[\Lambda_1(t - \tau \pm \pi)] + \mathrm{idem}\,(1 \leftrightarrow 2)$$

$$\Lambda_{1,2} = H \pm i\mu\Omega, \quad \Omega = \mathrm{diag}\,(\omega_1 E_{n_1}, \ldots, \omega_r E_{n_r})$$

Here $G(t,\tau)$ is Green's function of the boundary-value problem $y(0) = y(2\pi)$, $y^{\cdot}(0) = y^{\cdot}(2\pi)$ for (7.4); in the expressions for $G(t,\tau)$ and $G'(t,\tau)$, idem $(1 \leftrightarrow 2)$ denotes the summands obtained from the summands explicitly written out by the substitution $\Lambda_1 \leftrightarrow \Lambda_2$; the upper signs in these expressions are taken when $t \leqslant \tau$ and the lower when $t > \tau$. To derive formulas (7.5) we should notice that the matrices H, Ω, Λ_1 and Λ_2 commute with each other and the general solution of Eq. (7.4) with $f(t) = 0$ has the form $y = \exp(\Lambda_1 t)\, a_1 + \exp(\Lambda_2 t)\, a_2$, where a_1, a_2 are arbitrary constant vectors.

The derivative of solution (7.5) can be found from the formula

$$y^{\cdot}(t) = \int_0^{2\pi} \frac{\partial G(t,\tau)}{\partial t} f(\tau)\, d\tau = \int_0^{2\pi} G(t,\tau) f^{\cdot}(\tau)\, d\tau$$

The number

$$\|P\| = \Big(\sum_{j,k=1}^{l} |p_{jk}|^2\Big)^{1/2}$$

is called the norm of an arbitrary matrix $P = (p_{jk})_{j,k=1}^{l}$. Using this concept, we obtain the estimates

$$\nu(y) \leqslant K_1 \nu(f), \quad \nu(y^{\cdot}) \leqslant K_2 \nu(f)$$
$$\nu(y) \leqslant K_3 \nu(f^{\cdot}) + K_4 \nu(f), \quad \nu(y^{\cdot}) \leqslant K_1 \nu(f^{\cdot}) \tag{7.6}$$

for the norms of solution (7.5) and of its derivative. Here the coefficients K_1, K_2 and K_3 equal the maximum values, multiplied by 2π, of the functions $\| G(t, \tau)\|$, $\| \partial G(t, \tau)/\partial t\|$ and $\| G'(t, \tau)\|$ on the set $\{t, \tau : t, \tau \in [0,2\pi], \ t \neq \tau\}$, $K_4 = \| \Lambda_1^{-1}\Lambda_2^{-1}\|$. Let us estimate these coefficients. Since H and Ω are block diagonal matrices, the matrix $G(t, \tau)$ too is block diagonal with blocks of the same sizes and has the form

$$G(t, \tau) = \mathrm{diag}\,(G_1(t, \tau), \ldots, G_r(t, \tau))$$

where $G_j(t, \tau)$ is specified by the second formula in (7.5) with $\Lambda_{1,2} = H_j \pm i\mu\omega_j E_{n_j}$ $(j = 1, \ldots, r)$. It can be proved that

$$2\pi \max \| G_j(t, \tau) \| \leqslant \frac{d(n_j, H_j)}{\mu\omega_j \Delta(n_j, H_j, \mu\omega_j)}; \quad t, \tau \in [0, 2\pi]$$

Here $d(k, P)$ is some positive scalar function of the kth-order square matrix P. Since

$$\| G(t, \tau) \|^2 = \sum_{j=1}^{r} \| G_j(t, \tau) \|^2$$

for $\mu \in I(\varepsilon)$, $\mu > 0$, we have

$$K_1^2 \leqslant \frac{1}{\mu^4\varepsilon^2} \sum_{j=1}^{r} d_j, \quad d_j = \omega_j^{-2} d^2(n_j, H_j)$$

Analogously, for $\mu \in I(\varepsilon)$, $\mu > 0$, we can establish the estimates

$$K_2^2 \leqslant \frac{1}{\varepsilon^2} \sum_{j=1}^{r} d_j \| P_j \|^2, \quad K_3^2 \leqslant \frac{1}{\mu^4\varepsilon^2} \sum_{j=1}^{r} d_j \| P_j^{-1} \|^2, \quad K_4 \leqslant \frac{1}{\mu^2} \sum_{j=1}^{r} \| P_j^{-1} \|^2$$

where $P_j = \mu^{-1} H_j + i\omega_j E_{n_j}$. Since the matrix H_j is real, $\| P_j \|^2 = \omega_j^2 n_j + \mu^{-2} \| H_j \|^2$. Expanding the matrix P_j^{-1} in series in powers of μ^{-1}, we can prove that $\| P_j^{-1} \| \leqslant (\sqrt{n_j} + 1)/\omega_j$ for $\mu \geqslant 2\| H_j \|/\omega_j$. From the estimates indicated it follows that positive numbers N and μ_2 exist such that when $\mu \geqslant \mu_2$, $\mu \in I(\varepsilon)$, the inequalities $K_1 \leqslant N\mu^{-1}$, $K_2 \leqslant N$, $K_3 \leqslant N\mu^{-2}$, $K_4 \leqslant N\mu^{-2}$ are satisfied. From this and (7.6) we obtain the estimates

$$\nu(y) \leqslant N\mu^{-1}\nu(f), \quad \nu(y^{\cdot}) \leqslant N\nu(f) \tag{7.7}$$

$$\nu(y) \leqslant N\mu^{-2}(\nu(f) + \nu(f^{\cdot})), \quad \nu(y^{\cdot}) \leqslant N\mu^{-1}\nu(f^{\cdot}) \tag{7.8}$$

for the norms of solution (7.5) and of its derivative. Below, unless otherwise stated, it is assumed that $\mu \in I(\varepsilon)$ and $\mu \geqslant \mu_1 \geqslant \max(1, \mu_2)$.

8. The search for 2π-periodic solutions of system (6.9) reduces to solving the periodic boundary-value problem for this system in the interval $[0, 2\pi]$, which in turn is equivalent to the system of intergral equations

$$\xi(t) = \int_0^{2\pi} G_0(t, \tau) F_4(\tau, \xi(\tau), y_1(\tau), y_2(\tau), \mu)\, d\tau \equiv L_0(\xi, y_1, y_2) \tag{8.1}$$

$$y_j(t) = \int_0^{2\pi} g_j(t, \tau) [f_4(\tau, \xi(\tau), y_1(\tau), y_2(\tau), \mu) +$$
$$\mu^3 h_4(\tau, \xi(\tau), y_1(\tau), y_2(\tau), \mu)]\, d\tau \equiv L_j(\xi, y_1, y_2)$$
$$j = 1,2; \quad g_1(t, \tau) = G(t, \tau), \quad g_2(t, \tau) = \partial G(t, \tau)/\partial t$$

Here $y_1 = y$, $y_2 = y^{\cdot}$. System (8.1) is solved by the method of successive approximations. In the interval $0 \leqslant t \leqslant 2\pi$ we construct sequences of functions $\{\xi_k(t)\}_{k=0}^{\infty}$, $\{y_j^{(k)}(t)\}_{k=0}^{\infty}$ $(j = 1, 2)$, having set

$$\xi^{(0)}(t) \equiv 0, \quad y_j^{(0)}(t) \equiv 0 \tag{8.2}$$
$$\xi^{(k+1)} = L_0(\xi^{(k)}, y_1^{(k)}, y_2^{(k)}), \quad y_j^{(k+1)} = L_j(\xi^{(k)}, y_1^{(k)}, y_2^{(k)})$$
$$(j = 1, 2; \ k = 0, 1, 2, \ldots)$$

Let us prove that for fairly large μ these sequences converge to the solution of system (8.1).

First we will prove that positive numbers C_0, C_1, C_2 and μ_3 $(\mu_3 > \mu_1)$ exist such that the inequalities

$$\nu(\xi^{(k)}) \leqslant C_0\mu^{-1} \leqslant \delta, \quad \nu(y_1^{(k)}) \leqslant C_1\mu^{-2} \leqslant \delta$$
$$\nu(y_2^{(k)}) \leqslant C_2\mu^{-1} \leqslant \delta \quad (k = 0, 1, \ldots) \tag{8.3}$$

hold for $\mu > \mu_3$. The proof of this assertion is analogous to that of inequalities (4.3) and is carried out using the estimates (7.7), (7.8) and estimates (6.10) in the case when $\eta = 0,\ u = u^{\cdot} = 0$. We introduce the notation

$$a_k = \nu(\xi^{(k)} - \xi^{(k-1)}), \quad b_k = \nu(y_1^{(k)} - y_1^{(k-1)}), \quad c_k = \nu(y_2^{(k)} - y_2^{(k-1)})$$

On the strength of inequalities (6.10), (7.7) and (8.3), when $\mu > \mu_3$ we have

$$a_{k+1} \leqslant \varkappa\left(\frac{a_k}{\mu} + b_k + c_k\right), \quad b_{k+1} \leqslant \frac{\varkappa}{\mu}\left(\frac{a_k + c_k}{\mu} + b_k\right) \tag{8.4}$$
$$c_{k+1} \leqslant \varkappa\left(\frac{a_k + c_k}{\mu} + b_k\right) \quad (k = 1, 2, \ldots)$$
$$\varkappa = K[1 + 2(C_0 + C_1 + C_2)] \max(N_0,\ 2N)$$

Consider the number sequence $\rho_k = a_k\mu^{-1/2} + b_k + c_k\mu^{-1}$ $(k = 1, 2, \ldots)$. As a consequence of (8.4), $\rho_{k+1} \leqslant 3\varkappa\mu^{-1/2}\rho_k$ $(k = 1, 2, \ldots)$. We set $M = \max(\mu_3, 36\varkappa^2)$. Then when $\mu > M$ the estimate $\rho_{k+1} \leqslant \rho_k/2$ $(k = 1, 2, \ldots)$ is valid. Using this estimate it can be proved that the sequences $\{\xi^{(k)}(t)\}_{k=0}^{\infty}$, $\{y_j^{(k)}(t)\}_{k=0}^{\infty}$ $(j = 1, 2)$ converge uniformly in the interval $[0, 2\pi]$ to certain continuous functions $\xi^*(t)$, $y_j^*(t)$. The relations

$$\xi^*(0) = \xi^*(2\pi), \quad y_j^*(0) = y_j^*(2\pi) \quad (j = 1, 2)$$
$$\nu(\xi^*) \leqslant C_0\mu^{-1}, \quad \nu(y_1^*) \leqslant C_1\mu^{-2}, \quad \nu(y_2^*) \leqslant C_2\mu^{-1}$$

hold. Passing to the limit in (8.2) as $k \to \infty$, we find that $\xi^*(t)$, $y_1^*(t)$ and $y_2^*(t)$ is the solution of system (8.1). The function $\xi^*(t)$ is continuously differentiable, the function $y_1^*(t)$ is twice continuously differentiable, and $dy_1^*(t)/dt = y_2^*(t)$. Exactly as in the proof of Theorem 1 we can establish that the solution found is unique. Having continued the functions ξ^*, y_1^* 2π-periodically along the whole real axis, we obtain the desired periodic solution of system (6.9).

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THE HAMILTON-JACOBI EQUATION IN DOMAINS OF POSSIBLE MOTIONS WITH A BOUNDARY*

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The problem of the existence of solutions of the truncated Hamilton-Jacobi equation in the whole domain of possible motions with a boundary is investigated. Constraints on the topology of the domains of possible motions, in which the Hamilton-Jacobi equation is solvable in the large, are pointed out. In particular, the boundary cannot be connected. The existence of solutions in the whole domain of possible motions is obstructed by focal points at which infinitely close trajectories leaving the boundary intersect. A connection between the complete integral of the Hamilton-Jacobi equation and the particular solutions in the neighbourhood of the boundary is indicated.

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